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$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}\mathbf{u} = \Delta \mathbf{u} - \nabla p, \qquad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}(\cdot, 0) = \mathbf{u}_0.$$

$\mathbf{u} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ is the velocity field, p is the pressure and we assume for the viscosity $\nu = 1$ (always possible by rescaling).

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In spite of considerable progress, it is still unknown whether there are initial conditions for which the solution becomes singular in a finite time (global regularity problem).

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where a blowup is proved for a modified NS equations, which preserve the *energy identity*

$$E(t) + \int_0^t S(\tau)d\tau = E(0),$$

$E(t)$ is the total energy and $S(t)$ the total enstrophy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x}, \qquad S(t) = \int_{\mathbb{R}^3} |\nabla \times \mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x}.$$

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Poisson equation with Dirichlet boundary condition.

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$$\mathbf{u}(\mathbf{x}, t) = -i \int_{\mathbb{R}^3} e^{-i\langle \mathbf{k},\mathbf{x} \rangle} \mathbf{v}(\mathbf{k}, t) d\mathbf{k}$$

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$$-\nabla \cdot p(\mathbf{x}, t) = i \int_{\mathbb{R}^3} e^{-i\langle \mathbf{k},\mathbf{x}\rangle} \left(\int_{\mathbb{R}^3} \langle \mathbf{v}(\mathbf{k}-\mathbf{k}', t), \mathbf{k}' \rangle \frac{(\mathbf{k} \cdot \mathbf{v}(\mathbf{k}', t))}{|\mathbf{k}|^2} \mathbf{k} \, d\mathbf{k}' \right) d\mathbf{k}$$

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$$\int_{\mathbb{R}^3}\langle \mathbf{v}(\mathbf{k}-\mathbf{k}',t),\mathbf{k}\rangle P_{\mathbf{k}}\mathbf{v}(\mathbf{k}',t)d\mathbf{k}' = P_{\mathbf{k}}\int_{\mathbb{R}^3}\langle \mathbf{v}(\mathbf{k}-\mathbf{k}',t),\mathbf{k}\rangle \mathbf{v}(\mathbf{k}',t)d\mathbf{k}'$$

then

$$\mathbf{v}(\mathbf{k},t) = e^{-t\mathbf{k}^2}\mathbf{v}_0(\mathbf{k}) +$$

$$+\int_0^t e^{-(t-s)\mathbf{k}^2}\left(P_{\mathbf{k}}\int_{\mathbb{R}^3}\langle \mathbf{v}(\mathbf{k}-\mathbf{k}',s),\mathbf{k}\rangle\;\;\mathbf{v}(\mathbf{k}',s)\;d\mathbf{k}'\right)\;ds,$$

The integral equation is considered for real functions $\mathbf{v}(\mathbf{k},t)$.
The antitrasform $\mathbf{u}(\mathbf{x},t)$ is complex in general.

If however $\mathbf{v}(\mathbf{k},t)$ is odd in $\mathbf{k}$ then $\mathbf{u}(\mathbf{x},t)$ is real and odd in $\mathbf{x}$.

$$\int_{\mathbb{R}^3} \langle \mathbf{v}(\mathbf{k}-\mathbf{k}', s), \mathbf{k} \rangle \;\; v_j(\mathbf{k}', s) \; d\mathbf{k}' = \sum_{i=1}^{3} k_i \int_{\mathbb{R}^3} v_i(\mathbf{k}-\mathbf{k}', s) v_j(\mathbf{k}', s) \; d\mathbf{k}' =$$

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$$\mathbf{v}(\mathbf{k}, t+\Delta t) = e^{-(t+\Delta t)\mathbf{k}^2} \mathbf{v}_0(\mathbf{k}) + \int_0^{t+\Delta t} e^{-(t+\Delta t - s)\mathbf{k}^2} \mathbf{f}(\mathbf{v}, \mathbf{k}, s)\, ds =$$

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where $\beta_I = 1$, $\beta_{II} = \frac{1}{2}$ and $C_E^{(\alpha)}, C_S^{(\alpha)}$ are constants.

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iii) The solutions converge point-wise in $\mathbf{k}$-space as $t \uparrow \tau$, while $E(t)$ and $S(t)$ diverge as inverse powers of $\tau - t$.

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ii) the support of the solution goes away in the k_3 direction as $t \uparrow \tau$.

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We get solutions of type I for the initial data $\mathbf{v}_0^+$, and of type II (alternating signs) for $\mathbf{v}_0^-$.

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The mesh R in $\mathbf{k}$-space is taken with step 1:

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3. Li-Sinai solutions: simulations.

For the description of the behavior we use the energy and enstrophy marginals in **k** along the main axis

$$E_3(k_3, t) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}} dk_1 dk_2 |\mathbf{v}(\mathbf{k}, t)|^2,$$

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$$\tilde{E}_3(k_3,t)=\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} dx_1 dx_2 |\mathbf{u}(\mathbf{x},t)|^2,$$

$$\tilde{S}_3(k_3,t)=\int_{\mathbb{R}\times\mathbb{R}} dx_1 dx_2 |\nabla\mathbf{u}(\mathbf{x},t)|^2,$$

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- The solution of type I blow up much earlier than the solutions of type II with the same initial energy and same a.

3. Li-Sinai solutions: simulations. The fixed point $\mathbf{H}^{(0)}$.

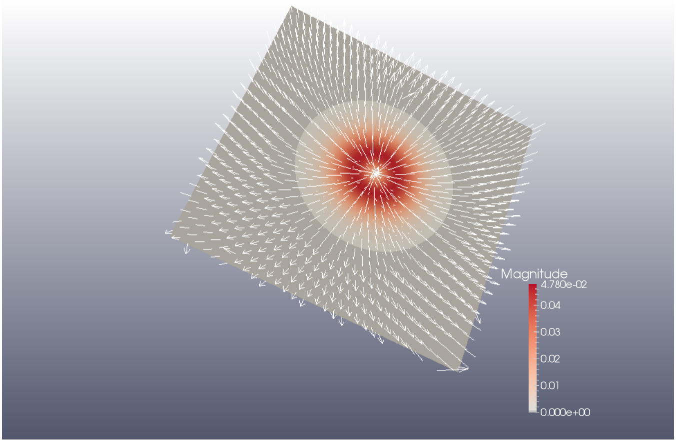

Figure 1: *Type II,* $a = 20$, $E_0 = 5 \times 10^4$. *The arrows indicate the direction of* $\mathbf{v}(\mathbf{k}, t)$ *on a regular point lattice on a section of the plane* $k_3 = 100$ *with sides of length* 100, $t = 1521 \times 10^{-7}$. *Simulation range* $k_3 \in [-19, 2528]$. *Magnitude refers to* $|\mathbf{v}(\mathbf{k}, t)|$. *In the grey external region* $|\mathbf{v}(\mathbf{k}, t)| < 10^{-6}$.

3. Li-Sinai solutions: simulations. Oscillations type I.

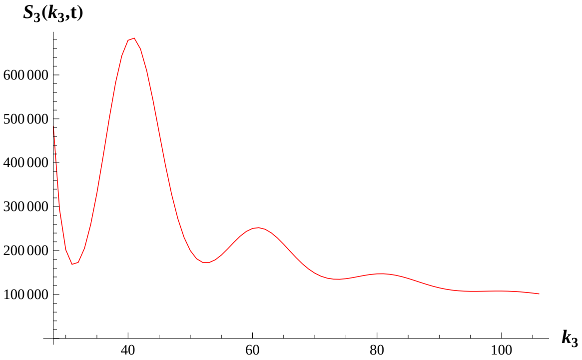

Figure 2: *Type I,* $a = 20$*,* $E_0 = 5 \times 10^4$*. Enstrophy marginal density* $S_3(k_3, t)$ *at the beginning of the blow-up.* $t = 900 \times 10^{-7}$*. Simulation range* $k_3 \in [-19, 2528]$*.*

3. Li-Sinai solutions: simulations. Oscillations type II.

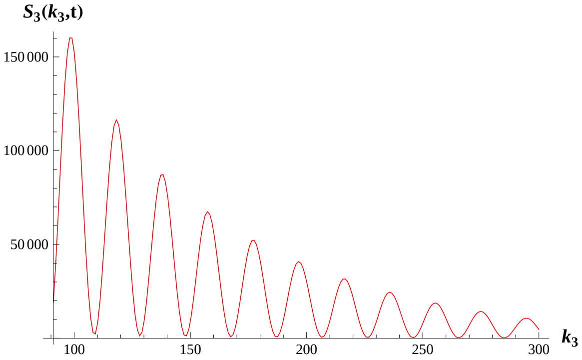

Figure 3: *Type II,* $a = 20$*,* $E_0 = 5 \times 10^4$*. Enstrophy marginal density* $S_3(k_3, t)$ *at the beginning of the blow-up.* $t = 1125 \times 10^{-7}$*. The zeroes are approximately periodic with period a. Simulation range* $k_3 \in [-19, 2528]$.

3. Li-Sinai solutions: simulations. Oscillations type II.

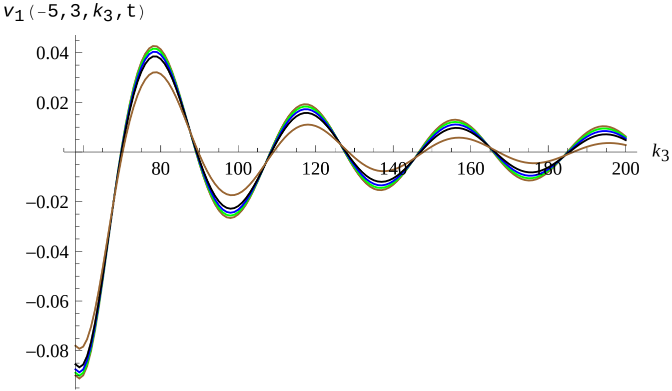

Figure 4: *Type II, $a = 20$, $E_0 = 5 \times 10^4$. $\mathbf{v}_1(\mathbf{k}, t)$ vs k_3 for k_1, k_2 fixed, at the times $t \times 10^7 = 1342, 1500, 1544, 1574, 1600$. The amplitudes increase as time grows, and tend to a limit. Simulation range $k_3 \in [-19, 3028]$.*

3. Li-Sinai solutions: simulations. Type I: compared growth.

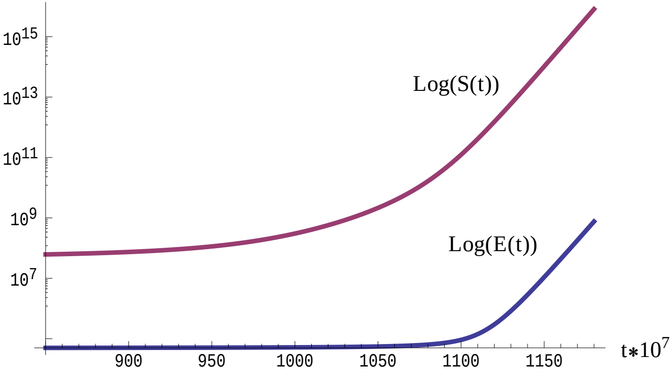

Figure 5: *Type I*, $a = 20$, $E_0 = 5 \times 10^4$. *Compared growth of the total enstrophy* $S(t)$ *and the total energy* $E(t)$. *Simulation range* $k_3 \in [-19, 2528]$.

3. Li-Sinai solutions: simulations. Type II: compared growth

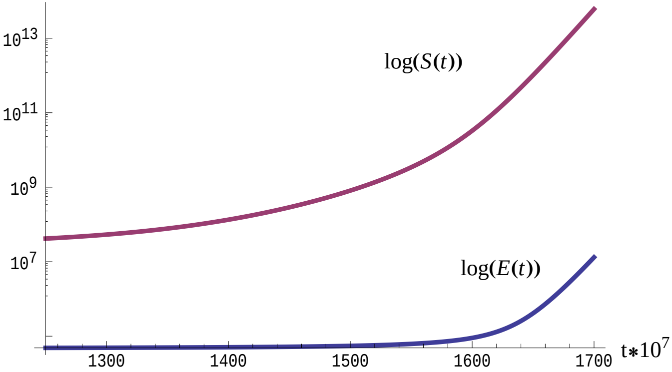

Figure 6: *Type II, $a = 20$, $E_0 = 5 \times 10^4$. Compared growth of the total enstrophy $S(t)$ and the total energy $E(t)$. Simulation range $k_3 \in [-19, 2528]$.*

3. Li-Sinai solutions: simulations. Enstrophy distribution

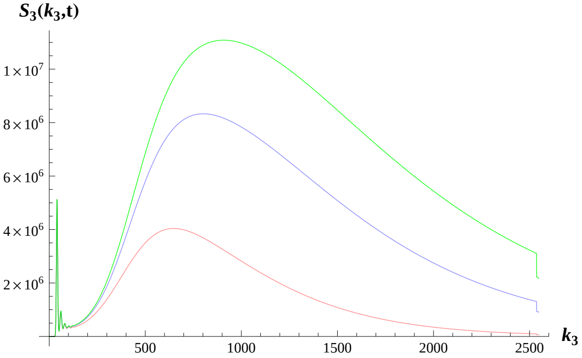

Figure 7: *Type I, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal enstrophy density $S_3(k_3, t)$ on the whole simulation range $-19 \leq k_3 \leq 2528$, at $t \cdot 10^7 = 1060, 1075, 1080$.*

3. Li-Sinai solutions: simulations. Enstrophy distribution

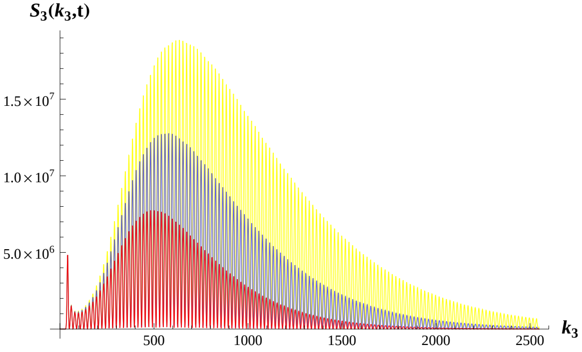

Figure 8: *Type II*, $a = 20$, $E_0 = 5 \times 10^4$. *Plot of the marginal enstrophy density* $S_3(k_3, t)$ *on the whole simulation range* $-19 \leq k_3 \leq 2528$, *at* $t \cdot 10^7 = 1521, 1544, 1560$.

3. Li-Sinai solutions: simulations. Enstrophy distribution.

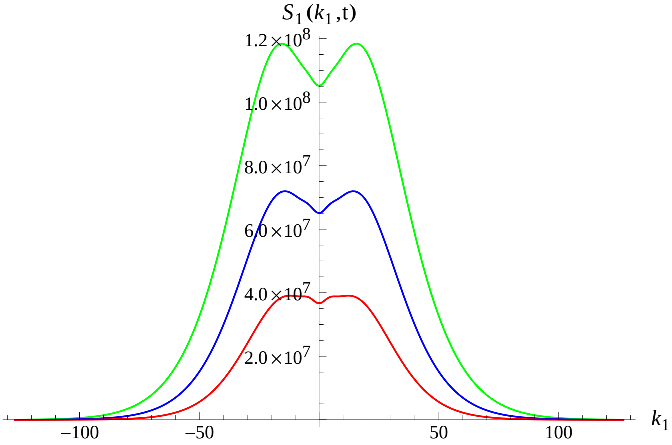

Figure 9: *Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal enstrophy density $S_1(k_1, t)$ at $t \cdot 10^7 = 1521, 1544, 1560$. Simulation range $k_3 \in [-19, 2528]$.*

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3. Li-Sinai solutions: simulations. Decay rate.

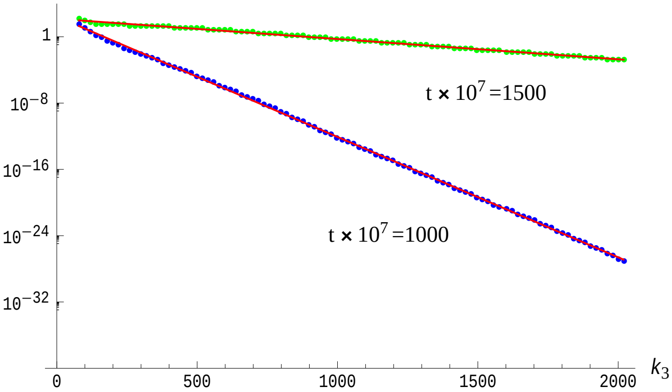

Figure 10: *Type II*, $a = 20$, $E_0 = 5 \times 10^4$. *Plot of* $\log(E_3(k_3, t))$, *where* E_3 *is the marginal energy density along the* k_3*-axis for* $k_3 \geq 400$ *at two different times. The dots represent the local maxima of the oscillations of* $E_3(k_3, t)$. *Simulation range* $k_3 \in [-19, 2028]$.

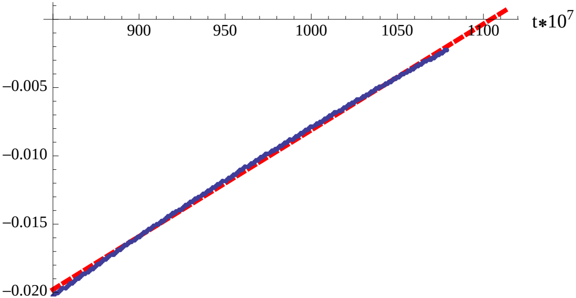

Figure 11: *Type I*, $a = 20$, $E_0 = 5 \times 10^4$. *Exponential decay rate for the marginal density* $E_3(k_3, t)$, *taken for* $k_3 \geq 400$, *vs magnified time* $t \times 10^7$, *with linear regression (dashed line). Simulation range* $k_3 \in [-19, 2528]$.

3. Li-Sinai solutions: simulations. Critical time.

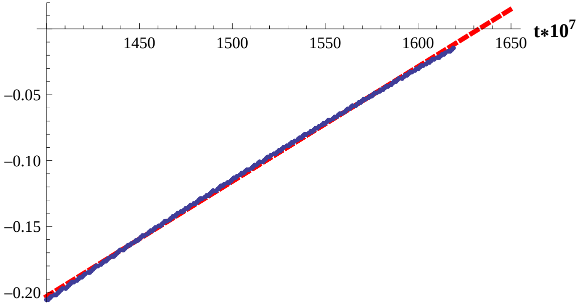

Figure 12: *Type II*, $a = 20$, $E_0 = 5 \times 10^4$. *Exponential decay rate for the marginal density* $E_3(k_3, t)$*, taken for* $k_3 \geq 400$*, vs magnified time* $t \times 10^7$*, with linear regression (dashed line). Simulation range* $k_3 \in [-19, 2528]$.

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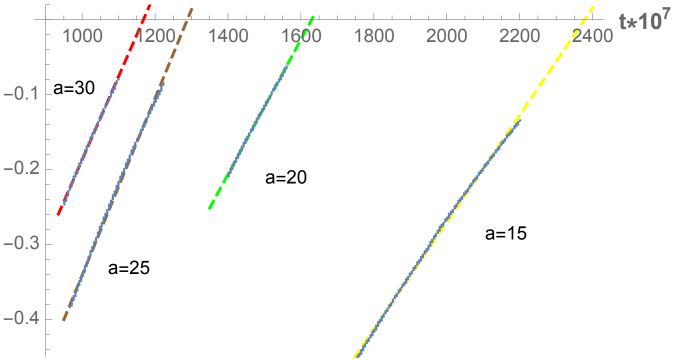

Figure 13: *Type II*, $E_0 = 5 \times 10^4$. *Behavior of the exponential decay rates of* $E_3(k_3, t)$ *vs. magnified time* $t \times 10^7$ *for* $a = 15, 20, 25, 30$. *Simulation range* $k_3 \in [-19, 3028]$

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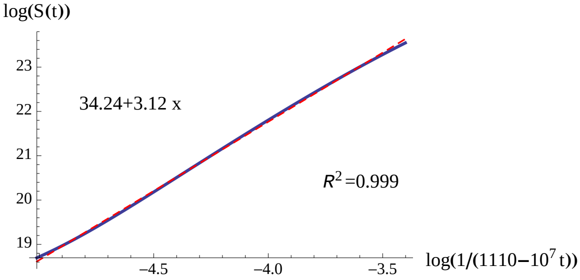

Figure 14: *Type I*, $a = 20$, $E_0 = 5 \times 10^4$. *Log-plot of the total enstrophy* $S(t)$ *vs* $\log \frac{1}{\tau_* - t}$, *at times near the blow-up, with linear regression (dashed line, the prediction for the slope is* 3.0*). Simulation range* $k_3 \in [-19, 2528]$.

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$$\mathbf{x}_{\pm}^{(0)} = (0, 0, \pm x_3^{(0)}), \qquad x_3^{(0)} \approx \frac{\pi}{a}$$

for the solutions of type II.

3. Li-Sinai solutions: simulations. x-space.

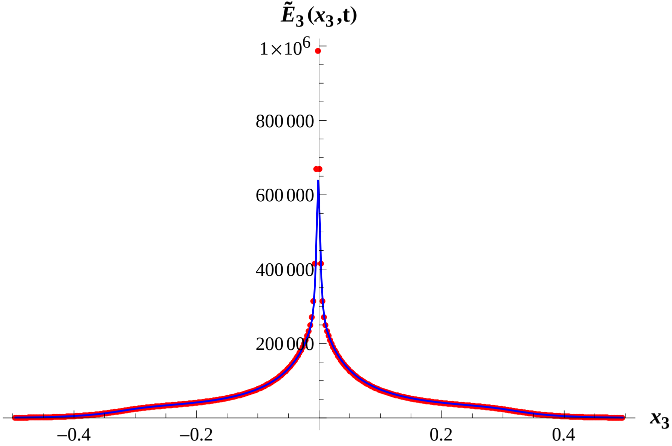

Figure 15: *Type I, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal energy density $\tilde{E}_3(x_3, t)$ at $t \cdot 10^7 = 1021$ (dotted line) and $t \times 10^7 = 1044$ (continuous line). Simulation range $k_3 \in [-19, 2528]$.*

3. Li-Sinai solutions: simulations. x-space.

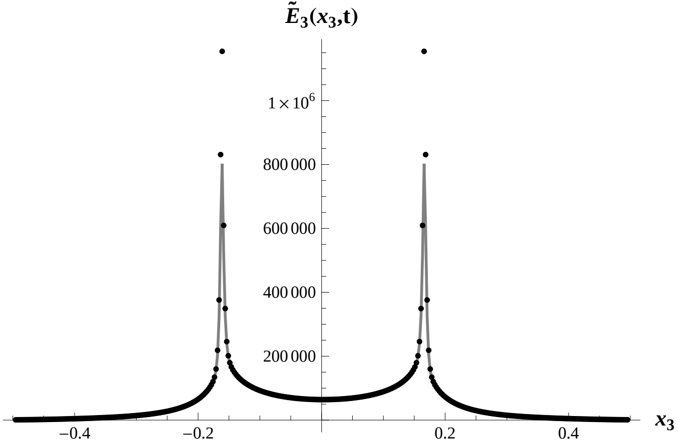

Figure 16: *Type II*, $a = 20$, $E_0 = 5 \times 10^4$. *Plot of the marginal energy density* $\tilde{E}_3(x_3,t)$ *at* $t \cdot 10^7 = 1521$ *(continuous line) and* $t \times 10^7 = 1544$ *(dotted line). Simulation range* $k_3 \in [-19, 2528]$

3. Li-Sinai solutions: simulations. **x**-space.

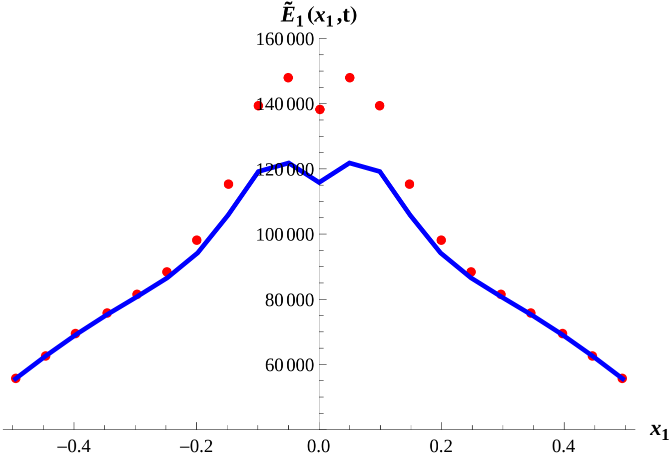

Figure 17: *Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal energy density $\tilde{E}_1(x_1, t)$ at $t \cdot 10^7 = 1521$ (continuous line), and $t \cdot 10^7 = 1544$ (dotted line). Simulation range $k_3 \in [-19, 2528]$.*

4. REAL SOLUTIONS.

Assuming antisymmetric initial data $A\mathbf{v}_0(\mathbf{k})$

$$\mathbf{v}_0(\mathbf{k}) = \mathbf{v}_0^{\pm}(\mathbf{k}) - \mathbf{v}_0^{\pm}(-\mathbf{k}),$$

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- The solution shows modulated oscillations along the k_3-axis, as the complex solutions of type II.

The simulation show, if the initial energy is large enough, that:

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- The total enstrophy $S(t)$ grows, reaches a maximum at a time t_* (which for a fixed depends on E_0), then falls;

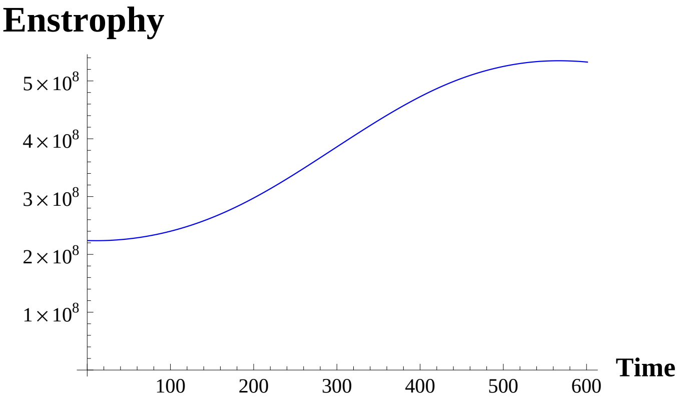

Figure 18: *Plot of the total enstrophy $S(t)$ vs. magnified time $t \times 3.2 \times 10^7$. Initial energy $\bar{E}_0 = 2.5 \times 10^4$, $a = 20$.*

- The large k_3 modes fall off exponentially fast, with a rate decreasing in absolute value up to $t \approx t_*$, then stays constant.

4. Real solutions.

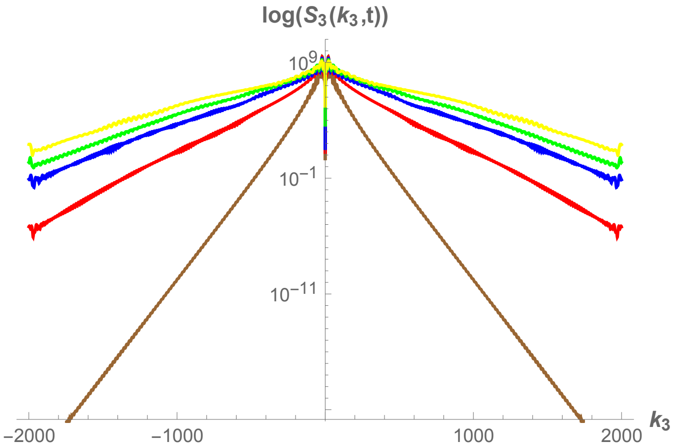

Figure 19: *Logarithmic plot of the marginal density* $\tilde{E}_3(k_3, t)$ *at the (magnified) times* $t \times 3.2 \times 10^7 = 100, 200, 300, 400, 500$. *Initial energy* $\bar{E}_0 = 2.5 \times 10^4$, $a = 20$.

- At the time t_*, the energy and the enstrophy concentrate in two (pseudo)-spikes, close to the singularities of the complex solution of type II with the same a.

4. Real solutions.

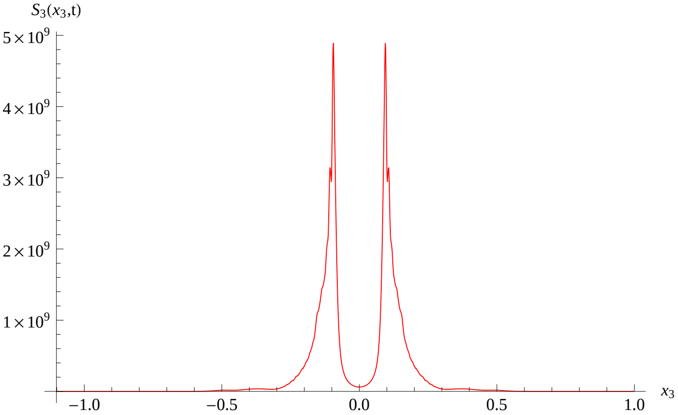

Figure 20: *Plot of the marginal density* $\tilde{S}_3(x_3, t)$ *at* $t = 1.27 \times 10^{-5}$. *Initial energy* $\bar{E}_0 = 2.5 \times 10^4$, $a = 20$.